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Dissipative chaotic quantum maps: Expectation values, correlation functions and the invariant state

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Abstract. I investigate the propagator of the Wigner function for a dissipative chaotic quantum map. I show that a small amount of dissipation reduces the propagator of sufficiently smooth Wigner functions to its classical counterpart, the Frobenius-Perron operator, if $\hbar \to 0$. Several consequences arise: the Wigner transform of the invariant density matrix is a smeared out version of the classical strange attractor; time dependent expectation values and correlation functions of observables can be evaluated *via* hybrid quantum-classical formulae in which the quantum character enters only *via* the initial Wigner function. If a classical phase-space distribution is chosen for the latter or if the map is iterated sufficiently many times the formulae become entirely classical, and powerful classical trace formulae apply.

PACS. 03.65.Sq Semiclassical theories and applications – 05.45.Mt Semiclassical chaos ("quantum chaos")

1 Introduction

This is the third of three articles in which I describe semiclassical methods for dissipative chaotic quantum maps. The first two articles [1,2] were devoted to spectral properties of such maps. It turned out that in the presence of a small amount of dissipation important information about the spectrum of the propagator can be obtained directly from the Frobenius-Perron propagator of the phase space density for the corresponding classical map. Spectral properties usually determine the behavior of most physical observables. It is therefore natural to ask what can be said about expectation values of observables and their correlation functions, which are the most interesting quantities from an experimental point of view.

In this article I lay a semiclassical framework that allows to calculate observables, correlation functions, and the invariant state of certain dissipative quantum maps. As a central result it turns out that the propagator of the Wigner function is to first order asymptotic expansion in $\hbar$ identical to the Frobenius-Perron propagator of the phase space density for the corresponding classical map. Many important consequences follow. In particular, time dependent expectation values and correlation functions are given by quantum classical hybrid formulae, in which the quantum character enters (to lowest order in $\hbar$) only *via* the initially prepared Wigner function. If this function is a classical phase space density, or after the map is iterated many times, the time dependent expectation values and correlation functions are given by entirely classical formulae. It follows that highly developed and precise classical periodic orbit theories can be applied.

There is currently very strong interest in understanding the interplay between chaos, dissipation and decoherence in quantum mechanics [3–15]. Dissipative quantum maps have been the object of choice in this area for a long time. The very same model as is studied here, namely a dissipative kicked top was investigated earlier numerically by Pepłowski *et al.*, and I am going to derive some of their results analytically. Particularly noteworthy is the study by Graham and Tél on the quantization of the Henon map [4]. They also examined the time evolution of the Wigner function and found by different means very similar results for the propagator as will be derived in the present work, namely a connection to the propagator of the entire *dissipative* classical map. Very similar results were also derived in the theory of superradiance, where the quantum character of the problem enters in an initial distribution of points in phase space, each then giving rise to a classical trajectory. In this way the macroscopically amplified quantum fluctuations of the delay time of the superradiant pulses was predicted [16].

The paper is structured as follows. After introducing in the next section the type of quantum maps that will be dealt with in this paper, I will derive in Section 3 a semiclassical approximation of the propagator of the Wigner function. Section 4 exploits the consequences for the invariant state, expectation values of observables, and correlation functions. The results are summarized in Section 5.

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2 Dissipative quantum maps

A dissipative quantum map P is a map of a density matrix ρ from a time t to a time $t+T$:

$$\rho(t+T) = P\rho(t). \tag{2.1}$$

The density matrix should be thought of as a reduced density matrix, resulting from a larger one, with the degrees of freedom of the environment which causes the dissipation traced out [17]. The type of maps that I consider in the following have been described in much detail in the earlier papers [1,2]. I will therefore restrict myself to introducing them here only very briefly, referring the interested reader to the above references.

The maps are particularly simple in the sense that the dissipation is well separated from a remaining purely unitary evolution where the latter by itself is capable of chaos. The unitary part is described by a unitary Floquet matrix F, and the dissipation by a propagator D. The prime example will be a dissipative kicked top, which is an angular momentum $\mathbf{J}$ whose orientation alters due to a continuous time evolution and periodic kicks. Between two unitary evolutions the angular momentum experiences damping. The dynamical variables of the top [18,19] are the three components $J_{x,y,z}$ of the angular momentum. Consider a unitary evolution generated by

$$F = \mathrm{e}^{-\mathrm{i}\frac{k}{2J}J_z^2}\mathrm{e}^{-\mathrm{i}\beta J_y}. \tag{2.2}$$

This Floquet matrix first rotates $\mathbf{J}$ by an angle β about the y-axis and then subjects it to a torsion, *i.e.* a rotation with a rotation angle proportional to J_z about the z-axis; the torsion strength is given by the parameter k. The maps considered (including the dissipative parts) have the important property that the square of $\mathbf{J}$ is conserved, $\mathbf{J}^2 = j(j+1) = const$ with j a fixed positive integer or half integer. They also have a well defined classical limit formally attained by $j \to \infty$. In (2.2) I have introduced $J = j + 1/2$ which simplifies most of the semiclassical formulae. It can be shown that the Bloch sphere $\lim_{j\to\infty} \mathbf{J}^2/J^2 = 1$ is the classical phase space [19]; $\mu = \cos\theta$ plays the role of classical momentum and ϕ the role of the conjugate coordinate. The angle θ denotes the polar angle of $\mathbf{J}$ reckoned against the J_z-axis, ϕ the azimuthal angle. The dynamics generated by F alone has been extensively studied in the literature [18,19] and is known to become strongly chaotic for sufficiently large values of k and β. The parameter values $k = 0$ or $\beta = 0$ lead to integrable motion.

As damping mechanism I consider a process which is given in continuous time τ by the Markovian master equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\rho(\tau) = \frac{1}{2J}([J_-, \rho(\tau)J_+] + [J_-\rho(\tau), J_+]) \equiv \Lambda\rho(\tau). \tag{2.3}$$

The linear operator Λ is hereby defined as generator of the dissipative motion. Equation (2.3) is well-known to describe certain superradiance experiments, where a large number of two-level atoms in a cavity of bad quality radiate collectively [20,21]. The angular momentum operator $\mathbf{J}$ is then the Bloch vector of the collective excitation and the J_+, J_- are raising and lowering operators, $J_\pm = J_x \pm \mathrm{i}J_y$. Equation (2.3) is formally solved by $\rho(\tau) = \exp(\Lambda\tau)\rho(0)$ for any initial density matrix $\rho(0)$, and this defines the dissipative propagator

$$D(\tau) = \exp(\Lambda\tau). \tag{2.4}$$

Explicit forms of D can be found in [20,22,23].

Damping manifests itself in a reduction of the J_z component which in the quantum optics application measures the energy stored in the two-level atoms. In the classical limit, formally attained by very large values j, the Bloch vector creeps towards the south pole $\theta = \pi$ of the Bloch sphere. The corresponding classical maps for rotation, torsion and dissipation can be found in the appendix of [2]. The time τ is measured in units of the classical time scale. In the following τ will be set equal to the time between two unitary steps, and therefore measures the dissipation strength. The total time will be measured in units of T, $t = NT$, and I will only keep track of the discrete time N.

Given F by (2.2) and D by (2.4), the total map reads

$$\rho(N+1) = D(F\rho(N)F^\dagger) \equiv P\rho(N). \tag{2.5}$$

I have suppressed the dependence on the system parameters k, β and τ. Due to the dissipation the total propagator P is non-unitary. It has always one eigenvalue equal to one which corresponds to an invariant density matrix. Its existence follows solely from probability conservation [19]. All other eigenvalues have absolute values smaller than one, reflecting the dissipative nature of the map. Correspondingly, the classical map leads to a shrinking phase space volume and, in the case of chaos, typically to strange attractors.

In the following a semiclassical approximation for P will be of importance which has been derived in [1]. I write it in the J_z-basis ($J_z|j,m\rangle = m|j,m\rangle$) and use indices m and k related to m_1, m_2 of $\rho_{m_1m_2}(N) = \langle j,m_1|\rho(N)|j,m_2\rangle \equiv \rho_m(k,N)$ by $m = (m_1+m_2)/2$ and $k = (m_1-m_2)/2$. In such a representation (2.5) reads

$$\rho_m(k,N+1) = \sum_{m'}\sum_{k'} P_{mk;m'k'}\rho_{m'}(k',N). \tag{2.6}$$

I convert the discrete sums into integrals by Poisson summation. For large J it is convenient to go over to rescaled coordinates, $\mu = m/J$ and $\eta = k/J$; the density matrix $\rho_m(k,N)$ will then be denoted by $\rho(\mu,\eta,N)$. Let us think of it as a continuous function of the continuous variables μ and η. Denoting the summation variables from the Poisson summation by s_1 and s_2, the new density matrix $\rho(\mu,\eta,N+1)$ in continuous arguments is obtained from

the old one, $\rho(\mu,\eta,N)$ according to

$$\rho(\mu,\eta,N+1) = 2J^2 \int \mathrm{d}\mu'\mathrm{d}\eta' \times \sum_{s_1,s_2=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2\pi J\left(s_1(\mu'+\eta')+s_2(\mu'-\eta')\right)} P(\mu,\eta;\mu',\eta')\rho(\mu',\eta',N). \tag{2.7}$$

Note that the Poisson summation was done in the original quantum numbers m_1', m_2' in order to avoid problems arising from the fact that m' and k' can be half integer. The factor two in front of the integral arises from the back transformation to m', k' and thus μ', η'.

Central object of interest for the subsequent study is the total propagator P. A semiclassical approximation has been derived in great detail in [1]. Here I just summarize the main features that will be of importance in the following and refer the reader interested in the details of the derivation to [1].

The semiclassical form of P is very reminiscent of a van Vleck propagator [24]. It is given by a double sum over all classical paths σ_1 and σ_2 joining the initial and final coordinates (μ',ϕ') and (μ,ϕ), respectively. The double sum arises because we propagate a density matrix and not a wave function. Along every path a complex "action" ψ is accumulated. It contains a real part R from the dissipative part of the motion and two imaginary components $\mathrm{i}S_{\sigma_1}$ and $-\mathrm{i}S_{\sigma_2}$ from the Floquet matrices F and $F^\dagger$,

$$\psi(\mu,\eta;\mu',\eta') = R(\mu,\overline{\nu};\eta) + \mathrm{i}\big(S_{\sigma_1}(\overline{\nu}+\eta,\mu'+\eta') - S_{\sigma_2}(\overline{\nu}-\eta,\mu'-\eta') + 2\pi l\overline{\nu}\big). \tag{2.8}$$

The dependence of ψ on its arguments arises from its implicit dependence on the intermediate coordinate $\overline{\nu} = \overline{\nu}(\mu,\eta;\mu',\eta')$ which is the solution of

$$\partial_{\overline{\nu}}\psi = 0. \tag{2.9}$$

The integer l stems from a Poisson summation that is still present in the propagator displayed below. The propagator also contains three pre-exponential factors, a function B from the dissipation, a C_{σ_1} from F and another $C^*_{\sigma_2}$ from $F^\dagger$. They are basically second mixed derivatives of the actions,

$$C_\sigma(\nu,\mu) = \frac{(-1)^j}{\sqrt{2\pi J}}\sqrt{|\partial_\nu\partial_\mu S_\sigma|}, \quad (\sigma=\sigma_1,\sigma_2), \tag{2.10}$$

$$B(\mu,\overline{\nu};\eta') = \sqrt{\frac{\partial\overline{\nu}}{\partial\mu}\left(\frac{\partial\overline{\nu}}{\partial\mu}\right)_{\tilde{E}}}. \tag{2.11}$$

Both derivatives in the last equation have to be taken at constant η, and the second one additionally at constant energy $\tilde{E}$ of a fictitious Hamiltonian system that underlies the dissipative dynamics [23]. These distinctions will be of no further importance in the remainder, though, as the saddle point approximations that will come up soon, pick classical trajectories at $\eta = 0$. For these both derivatives under the square root are the same and both combine to the Jacobian connected with the classical trajectory [1].

In its full-fledged form the propagator reads

$$P(\mu,\eta;\mu',\eta') = \sum_{l=-\infty}^{\infty}\sum_{\sigma_1,\sigma_2}\sum_{\overline{\nu}} B(\mu,\overline{\nu};\eta) \times C_{\sigma_1}(\overline{\nu}+\eta,\mu'+\eta')C^*_{\sigma_2}(\overline{\nu}-\eta,\mu'-\eta') \times \exp\left(J\psi(\mu,\eta;\mu',\eta')\right). \tag{2.12}$$

The explicit form of the actions S and R is not relevant here. More important are their generating properties for the classical trajectories,

$$\partial_\mu S_\sigma(\nu,\mu) = \phi^{\mathrm{i}}_\sigma(\nu,\mu), \quad \partial_\nu S_\sigma(\nu,\mu) = -\phi^{\mathrm{f}}_\sigma(\nu,\mu), \tag{2.13}$$

where ϕ^{i}_σ and ϕ^{f}_σ are the initial and final coordinates of the classical trajectory σ corresponding to the unitary part of the map ($\sigma=\sigma_1,\sigma_2$). For R we have

$$\partial_\mu R(\mu,\nu;0) = 0 \;\Leftrightarrow\; \mu = \mu_d(\nu) \;\Leftrightarrow\; \partial_\nu R(\mu,\nu;0) = 0, \tag{2.14}$$

where $\mu = \mu_d(\nu)$ denotes the (unique) classical trajectory corresponding to the dissipative part of the motion. Furthermore, R as a function of η has a single maximum at $\eta = 0$, *i.e.*

$$\partial_\eta R(\mu,\nu;\eta) = 0 \Leftrightarrow \eta = 0. \tag{2.15}$$

3 Wigner function and Wigner propagator

In order to unravel classical properties of the quantum map, it is natural to go to a phase space representation of the density matrix. It turns out that the Wigner function is very well suited for this purpose. In fact, the Wigner function has been used many times in order to study the transition from quantum to classical mechanics [3–9,11,12]. I will show in this section that the propagator of the Wigner function is – for sufficiently smooth Wigner functions – nothing but the classical Frobenius-Perron propagator of the phase space density.

Usually the Wigner transform is defined as a Fourier transform with respect to the skewness of the density matrix in coordinate representation [13,25],

$$\rho_{\mathrm{W}}(p,q) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{\mathrm{i}px/\hbar}\langle q-\frac{x}{2}|\hat{\rho}|q+\frac{x}{2}\rangle. \tag{3.1}$$

In our problem we have ρ in the momentum basis μ. Inserting resolutions of the identity operator in the momentum basis in the above definition of $\rho_{\mathrm{W}}(p,q)$ we obtain

$$\rho_{\mathrm{W}}(p,q) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{\mathrm{i}q\xi/\hbar}\langle p+\frac{\xi}{2}|\hat{\rho}|p-\frac{\xi}{2}\rangle. \tag{3.2}$$

Note the change of sign in the skewness. We have $p = \mu$, $q = \phi$, and $1/J$ replaces $\hbar$. An additional factor J arises because the original quantum numbers m and k are rescaled to μ and η as explained above. I therefore define the Wigner transform of $\rho(\mu, \eta, N)$ as

$$\rho_{\rm W}(\mu, \phi, N) = \frac{J^2}{2\pi} \int {\rm d}\eta \, {\rm e}^{{\rm i}J\eta\phi} \rho(\mu, \frac{\eta}{2}, N). \tag{3.3}$$

It has the right normalization in the sense that

$$\int {\rm d}\mu {\rm d}\phi \, \rho_{\rm W}(\mu, \phi, N) = J \int {\rm d}\mu \, \rho(\mu, 0, N) \simeq \sum_m \rho_{mm}(N) = 1. \tag{3.4}$$

Note that the corrections from passing from the integral to the discrete sum of the diagonal matrix elements are of order $1/J$ and become negligible in the limit of large J, as long as $\rho(\mu, 0, N)$ does not fluctuate on a scale $1/J$, *i.e.* as long as the probability profile has a classical meaning.

The inverse transformation reads

$$\rho(\mu, \eta, N) = \frac{1}{J} \int {\rm d}\phi \, {\rm e}^{-{\rm i}2J\eta\phi} \rho_{\rm W}(\mu, \phi, N). \tag{3.5}$$

Wigner functions on $SU(2)$ have been introduced before in the literature [26,27] *via* angular momentum coherent states and appropriate transformations of Q- or P-functions. These definitions avoid problems at the poles of the Bloch sphere that can arise in the present approach. On the other hand, the definition (3.3) is much simpler from a technical point of view and sufficient for our purposes.

We are now in the position to calculate the Wigner function after one application of the map from the original one. To this end we insert the propagated density matrix $\rho(\mu, \eta, N+1)$ from (2.7) in

$$\rho_{\rm W}(\mu, \phi, N+1) = \frac{J^2}{\pi} \int_{-\infty}^{\infty} {\rm d}\eta \, {\rm e}^{2{\rm i}J\eta\phi} \rho(\mu, \eta, N+1) \,, \tag{3.6}$$

and then express the original density matrix $\rho(\mu', \eta', N)$ in terms of its Wigner transform,

$$\rho_{\rm W}(\mu, \phi, N+1) = \frac{2J^3}{\pi} \int {\rm d}\eta {\rm d}\mu' {\rm d}\eta' {\rm d}\phi' \times \sum_{s_1, s_2} \exp\left(2{\rm i}J(\pi((s_1+s_2)\mu' + (s_1-s_2)\eta') - \eta'\phi' + \eta\phi)\right) \times P(\mu, \eta; \mu', \eta') \, \rho_{\rm W}(\mu', \phi', N). \tag{3.7}$$

With the semiclassical expression (2.12) for the propagator, we arrive at

$$\rho_{\rm W}(\mu, \phi, N+1) = \frac{2J^3}{\pi} \int {\rm d}\eta {\rm d}\mu' {\rm d}\eta' {\rm d}\phi' \times \sum_{\sigma_1, \sigma_2, s_1, s_2, \overline{\nu}, l} B(\mu, \overline{\nu}; \eta) C_{\sigma_1}(\overline{\nu} + \eta, \mu' + \eta') C^*_{\sigma_2}(\overline{\nu} - \eta, \mu' - \eta') \times \exp\left(JG(\mu, \eta; \mu', \eta')\right) \, \rho_{\rm W}(\mu', \phi', N), \tag{3.8}$$

where the "action" G is given by

$$G(\mu, \eta; \mu', \eta') = \psi(\mu, \eta; \mu', \eta') + 2\eta\phi - 2\eta'\phi' + 2\pi(s_1(\mu' + \eta') + s_2(\mu' - \eta')). \tag{3.9}$$

The form of the integrands and the fact that P is already approximated semiclassically, *i.e.* correct only to order $1/J$, suggests to integrate by saddle point approximation (SPA). To do so, we must assume that the initial Wigner function $\rho_{\rm W}(\mu', \phi')$ is sufficiently smooth, *i.e.* has no structure on a scale $1/J$. This is at the same time a necessary condition if we want to attribute a classical meaning to $\rho_{\rm W}$.

The saddle point equations read

$$\partial_\eta G = \partial_\eta R + \partial_{\overline{\nu}} R \partial_\eta \overline{\nu} + {\rm i}\big[\big(-\phi^{\rm f}_{\sigma_1} + \phi^{\rm f}_{\sigma_2} + 2\pi l\big)\partial_\eta \overline{\nu} - \phi^{\rm f}_{\sigma_1} - \phi^{\rm f}_{\sigma_2} + 2\phi\big] = 0, \tag{3.10}$$

$$\partial_{\mu'} G = \partial_{\overline{\nu}} R \partial_{\mu'} \overline{\nu} + {\rm i}\big[\big(-\phi^{\rm f}_{\sigma_1} + \phi^{\rm f}_{\sigma_2} + 2\pi l\big)\partial_{\mu'} \overline{\nu} + \phi^{\rm i}_{\sigma_1} - \phi^{\rm i}_{\sigma_2} + 2\pi(s_1 + s_2)\big] = 0, \tag{3.11}$$

$$\partial_{\eta'} G = \partial_{\overline{\nu}} R \partial_{\eta'} \overline{\nu} + {\rm i}\big[\big(-\phi^{\rm f}_{\sigma_1} + \phi^{\rm f}_{\sigma_2} + 2\pi l\big)\partial_{\eta'} \overline{\nu} + \phi^{\rm i}_{\sigma_1} + \phi^{\rm i}_{\sigma_2} + 2\pi(s_1 - s_2) - 2\phi'\big], \tag{3.12}$$

$$\partial_{\phi'} G = 2{\rm i}\eta' = 0. \tag{3.13}$$

For brevity I have suppressed the arguments of R and $\phi^{\rm i}_\sigma, \phi^{\rm f}_\sigma$. They are the same as in (2.8) for R and S_σ, $\sigma = \sigma_1, \sigma_2$. Equation (3.13) immediately gives $\eta' = 0$. To solve the rest of the equations, let us first assume that $\partial_{\overline{\nu}} R = 0$. I will show below that this is the only possible choice. Then we have from the generating property (2.14) that μ is connected to $\overline{\nu}$ *via* the classical dissipative trajectory, $\mu = \mu_d(\overline{\nu})$ and the real parts of (3.11, 3.12) give already zero. I will assume that all relevant solutions to the saddle point equation be real, as is expected from the physical origin of the variables as real valued quantum numbers. Real- and imaginary-parts of all saddle point equations must then separately equal zero, so that we confront eight instead of four equations. The assumption $\partial_{\overline{\nu}} R = 0$ solves two of them at the same time. The real part from (3.10) gives additionally $\partial_\eta R = 0$ and thus according to the property (2.15) $\eta = 0$. Only the propagation of probabilities, *i.e.* the diagonal elements of the density matrix contributes in the saddle point approximation; the same was true for the calculation of ${\rm tr} P^N$ [1].

From (2.9) follows ${\rm i}\big(-\phi^{\rm f}_{\sigma_1} + \phi^{\rm f}_{\sigma_2} + 2\pi l\big) = 0$, *i.e.* $-\phi^{\rm f}_{\sigma_1}(\overline{\nu}, \mu) + \phi^{\rm f}_{\sigma_2}(\overline{\nu}, \mu) + 2\pi l = 0$. Thus, the final canonical coordinates of the two trajectories σ_1 and σ_2 must agree up to integer multiples of 2π. Since also initial and final momenta are the same (μ and $\overline{\nu}$, respectively), the two trajectories must be identical, *i.e.* $\sigma_1 = \sigma_2 \equiv \sigma$. Counting all angles modulo 2π we also have $l = 0$.

The imaginary part of (3.10) leads to $\phi^{\rm f}_\sigma = \phi$, the imaginary part of (3.11) to $s_1 + s_2 = 0$, and the imaginary

part of (3.12) to $\phi^{\rm i}_\sigma = \phi' + 2\pi s_2$. These equations describe precisely the classical trajectories for the unitary part of the motion from an initial phase space point (μ', ϕ') to a final one $(\overline{\nu}, \phi)$, again counting the angle modulo 2π. Together with $\mu = \mu_d(\overline{\nu})$ the saddle point equations thus give the classical trajectory from (μ', ϕ') to (μ, ϕ). Note that this trajectory is unique if it exists, since classical trajectories are uniquely defined by their starting point in phase space.

For evaluating the SPA we further need the determinant of the matrix $G^{(2)}$ of second derivatives of G. It is straightforward to verify that its absolute value is given by

$$|\det G^{(2)}| = 16|\partial_{\overline{\nu}}\phi^{\rm i}_\sigma(\overline{\nu}, \mu')|^2. \tag{3.14}$$

The overall phase arising from the SPA equals zero, as can be seen by the same techniques that were used in [1]. Calling the classical map f: $(\mu', \phi') \to (\mu, \phi) = f(\mu', \phi')$, we get the saddle point approximation

$$\begin{aligned}\rho_{\rm W}(\mu, \phi, N+1) &= \frac{2J^3}{\pi}\sqrt{\frac{(2\pi)^4}{J^4|\det G^{(2)}|}}B(\mu_d(\overline{\nu}), \overline{\nu}; 0) \\ &\quad \times |C(\overline{\nu}, \mu')|^2 \rho_{\rm W}(f^{-1}(\mu, \phi), N) \qquad (3.15)\\ &= \left|\frac{\partial\overline{\nu}}{\partial\mu}\right| \rho_{\rm W}(f^{-1}(\mu, \phi), N). \qquad (3.16)\end{aligned}$$

The prefactor in the last equation is nothing but the inverse of the Jacobian of the classical trajectory which arises solely from the dissipative step since the unitary one has Jacobian unity. So with the abbreviation $y = (\mu, \phi)$, $x = (\mu', \phi')$ of final and initial phase space coordinates we have

$$\begin{aligned}\rho_{\rm W}(y, N+1) &= \frac{\rho_{\rm W}(f^{-1}(y), N)}{|\partial f/\partial x|_{x=f^{-1}(y)}} \\ &= \int {\rm d}x\, \delta(y - f(x))\rho_{\rm W}(x, N) \\ &\equiv \int {\rm d}x\, P_{\rm W}(y, x)\rho_{\rm W}(x, N), \qquad (3.17)\end{aligned}$$

which identifies the propagator of the Wigner function as the classical Frobenius-Perron propagator of the phase space density,

$$P_{\rm W}(y, x) = P_{\rm cl}(y, x) = \delta(y - f(x)). \tag{3.18}$$

Note once more that this conclusion holds only if the test function $\rho_{\rm W}(x)$ on which $P_{\rm W}$ acts is sufficiently smooth, namely does not contain any structure on a scale $1/J$ or smaller. For classical phase space densities this is often not the case. Indeed, continued application of a chaotic map leads to ever finer phase space structure, so that after the Ehrenfest time of order $\lambda^{-1}\ln J$ (where λ is the largest Lyapunov exponent) scales are reached that are comparable with $1/J$. From equation (3.18) it follows immediately that also the propagators of the iterated maps are identical, $P^N_{\rm W} = P^N_{\rm cl}$, but the validity is restricted to discrete times N much smaller than the Ehrenfest time. These restrictions not withstanding, (3.18) is of rather general validity, as for the derivation only the van Vleck forms of the dissipative and unitary propagators and the generating properties of the actions involved were used. Dissipation can be arbitrarily strong, but should not become smaller than $\tau \simeq 1/J$ since otherwise the van Vleck form of the propagator for equation (2.3) breaks down.

Let me finally show that there is no alternative to the assumption $\partial_{\overline{\nu}}R = 0$ about the solution of the saddle point equation if only real solutions are permitted. To see this suppose that $\partial_{\overline{\nu}}R \neq 0$. Then we have from (3.11) that $\partial\overline{\nu}/\partial\mu' = 0$, and from (3.12) $\partial\overline{\nu}/\partial\eta' = 0$, such that $\overline{\nu}$ is a function of μ and η alone. The imaginary part of (3.11) gives $\phi^{\rm i}_{\sigma_1} - \phi^{\rm i}_{\sigma_2} + 2\pi(s_1 + s_2) = 0$. If we differentiate with respect to μ' and with respect to η' and remember that $\eta' = 0$ follows directly from (3.13), we are immediately lead to $\partial_{\mu'}\phi^{\rm i}_{\sigma_1}(\overline{\nu} + \eta, \mu') = \partial_{\mu'}\phi^{\rm i}_{\sigma_2}(\overline{\nu} - \eta, \mu') = 0$. Thus, all trajectories with given initial $\phi^{\rm i}_\sigma$ end at the same final momentum $\overline{\nu} + \eta$ (for $\sigma = \sigma_1$) or $\overline{\nu} - \eta$ (for $\sigma = \sigma_2$). From the imaginary part of (3.10) follows in the same fashion $\partial_{\mu'}\phi^{\rm f}_{\sigma_1}(\overline{\nu} + \eta, \mu') = \partial_{\mu'}\phi^{\rm f}_{\sigma_2}(\overline{\nu} - \eta, \mu') = 0$. So the final canonical coordinate does not depend on the initial momentum either. In other words, all trajectories with the same initial $\phi^{\rm i}_{\sigma_1}$ (respectively $\phi^{\rm i}_{\sigma_2}$) but arbitrary initial μ' end at the same final phase space point. But this is in contradiction with the fact that a final phase space point uniquely defines a trajectory. Therefore, the initial assumption $\partial_{\overline{\nu}}R \neq 0$ must be wrong.

4 Consequences

Equation (3.18) is a key equation from which many consequences follow in a straightforward way. Let me first show that previous results about spectral properties are readily recovered.

4.1 Spectral properties

In [1,2] we have shown with great effort that

$${\rm tr}P^N = {\rm tr}P^N_{\rm cl} \tag{4.1}$$

for all integer N, if $J \to \infty$. The same result can now be obtained much more easily. In view of (3.18) all that remains to do is to show that ${\rm tr}P^N = {\rm tr}P^N_{\rm W}$. To see this, let us extract the general relation between any $P_{\rm W}$ and the corresponding P from (3.7) and the definition of $P_{\rm W}$ in (3.17). Comparing the two equations we are lead to

$$\begin{aligned}P_{\rm W}(\mu, \phi; \mu', \phi') &= \frac{2J^3}{\pi}\int {\rm d}\eta\, {\rm d}\eta' \\ &\quad \times \sum_{s_1, s_2} {\rm e}^{2{\rm i}J(\eta'\phi' - \eta\phi) + {\rm i}2\pi J\left((s_1+s_2)\mu + (s_1-s_2)\eta'\right)} P(\mu, \eta; \mu', \eta').\end{aligned} \tag{4.2}$$

The equation holds for any propagator P of the density matrix, therefore also for the propagator P^N of the Nth

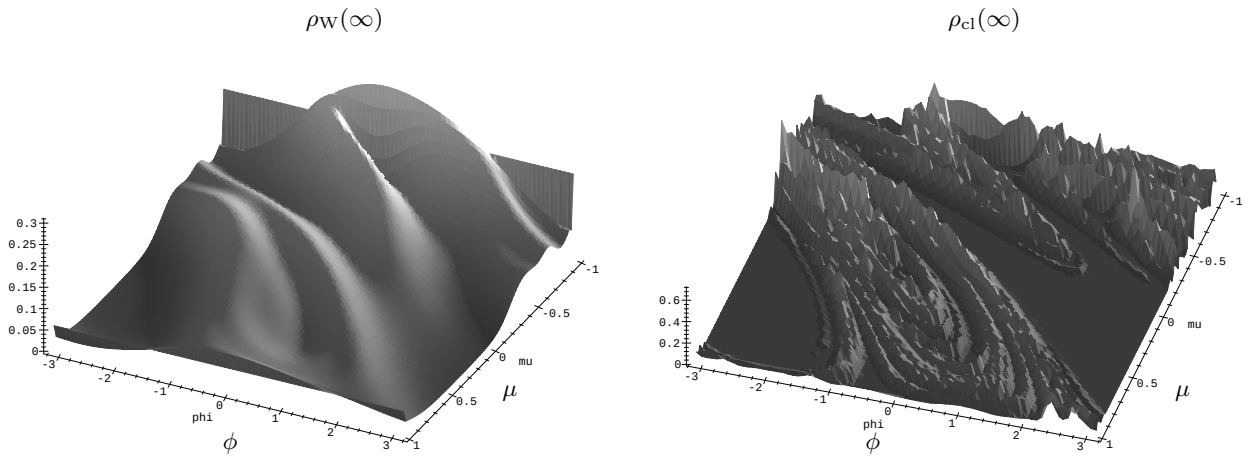

Fig. 1. Wigner function $\rho_{\rm W}(\infty)$ corresponding to the invariant density matrix ($j = 40$) and classical stationary probability distribution $\rho_{\rm cl}(\infty)$ (strange attractor) for $k = 4.0$, $\beta = 2.0$, $\tau = 0.5$. The Wigner function is a "quantum strange attractor", a smeared out version of the classical strange attractor. The range of arguments is $\phi = -\pi \ldots \pi$ and $\mu = -1$ (in the background) $\ldots 1$ (in the foreground).

iteration of the original map. It is then one line of calculation to show that the trace of $P_{\rm W}^N$,

$$\mathrm{tr}P_{\rm W}^N = \int \mathrm{d}\mu\, \mathrm{d}\phi\, P_{\rm W}^N(\mu,\phi;\mu,\phi), \tag{4.3}$$

is given by

$$\mathrm{tr}P_{\rm W}^N = 2J^2 \int \mathrm{d}\eta\, \mathrm{d}\mu \sum_{s_1,s_2} \mathrm{e}^{\mathrm{i}2\pi J((s_1+s_2)\mu+(s_1-s_2)\eta)} \times P^N(\mu,\eta;\mu,\eta) = \mathrm{tr}P^N, \tag{4.4}$$

where in the last equation I have gone back to discrete summation, undoing the Poisson summation. Thus, we have, up to $\mathcal{O}(1/J)$ corrections, $\mathrm{tr}P^N = \mathrm{tr}P_{\rm W}^N = \mathrm{tr}P_{\rm cl}^N$.

4.2 The invariant state

If the classical and the Wigner propagator are the same up to order $1/J$ corrections, so are their eigenstates. An invariant state of P is defined as an eigenstate with eigenvalue one, $P\rho(\infty) = \rho(\infty)$, and correspondingly $P_{\rm W}\rho_{\rm W}(\infty) = \rho_{\rm W}(\infty)$, $P_{\rm cl}\rho_{\rm cl}(\infty) = \rho_{\rm cl}(\infty)$. Classically, there can be many invariant states. For example a distribution $\rho_{\rm cl}(x) = \delta(x - x_{\rm fp})$ is an invariant distribution if $x_{\rm fp}$ is a fixed point in the neighborhood of which the map preserves area. If several fixed points exist one can linearly combine the delta functions on them to obtain new invariant distributions. In the present context we are interested, however, in invariant distributions that are not only stationary, but can also be obtained from a general initial state by iterating the map infinitely many times. I indicate this with the arguments "infinity". If the system is ergodic, the final invariant distribution is unique. For volume preserving chaotic maps, it is a homogeneous distribution in phase space regions selected by the remaining integrals of motion. For dissipative chaotic maps one encounters typically a strange attractor in phase space [28].

From (3.18) we conclude that up to $\mathcal{O}(1/J)$ corrections

$$\rho_{\rm W}(\infty) = \rho_{\rm cl}(\infty). \tag{4.5}$$

The corrections have to be understood as a smearing out on a scale $1/J$. Indeed, suppose we start from a smooth initial Wigner function, and then iterate it many times with $P_{\rm W}$, it evolves according to (3.18) up to the Ehrenfest time as a classical phase space density. After the Ehrenfest time the classical dynamics continues to produce ever finer structures in the phase space density, whereas Heisenberg's uncertainty principle prohibits structures of $\rho_{\rm W}$ smaller than $1/J$. As pointed out before, (3.18) therefore ceases to be valid, and $\rho_{\rm W}$ is left at the stage where it is the smeared out classical strange attractor. Figure 1 shows that indeed the quantum strange attractor is a smeared out classical one. The Wigner function was obtained by direct diagonalization of the propagator and subsequent Wigner transformation of the eigenstate with eigenvalue unity, the classical picture by iterating classical trajectories many times and making a histogram in phase space. Similar observations were made earlier numerically [5,13,29,30].

4.3 Expectation values

Suppose that a system is prepared at time $t = 0$ by specifying the density matrix $\rho(0)$, or equivalently the initial Wigner function $\rho_{\rm W}(x,0)$. We let the system evolve for a discrete time N and then measure any observable $\hat{A}$ of interest. The expectation value of the observable is given by

$$\langle A(N)\rangle \equiv \mathrm{tr}(\hat{A}\rho(N)) = \int \mathrm{d}x\, A_{\rm W}(x)\rho_{\rm W}(x,N)\,, \tag{4.6}$$

where $A_{\rm W}(x)$ is the Weyl symbol associated with the operator $\hat{A}$. The definition of A is analogous to the definition of $\rho_{\rm W}$ [13]. So $A_{\rm W}(x)$ is also a phase space function.

To lowest order in $1/J$ it equals the classical observable $A(x)$ that corresponds to $\hat{A}$, if the classical observable exists. Using (3.18) we immediately obtain

$$\langle A(N)\rangle = \int \mathrm{d}x\, A(x) P_{\mathrm{cl}}^N \rho_{\mathrm{W}}(x,0)\,, \tag{4.7}$$

up to corrections of order $1/J$. Thus, quantum mechanical expectation values can be obtained from the knowledge of the classical propagator and the classical observable for any initial Wigner function that contains no structure on the scale $1/J$. Equation (4.7) is a hybrid classical-quantum formula, since the initial Wigner function can be very *non-classical*, *e.g.* can contain regions where $\rho_{\mathrm{W}}(x,0) < 0$.

But also the cases are interesting where already the initial Wigner function is a classical phase space density, $\rho_{\mathrm{W}} = \rho_{\mathrm{cl}}$, or where the time N is sent to infinity, such that the invariant state is reached. In both cases the quantum mechanical expectation value is given by a purely classical formula. In particular, expectation values in the invariant state $\rho_{\mathrm{W}}(\infty)$ are given by

$$\langle A\rangle_\infty = \int \mathrm{d}x\, A(x)\rho_{\mathrm{cl}}(x,\infty)\,, \tag{4.8}$$

since up to order $1/J$ corrections $P_{\mathrm{cl}}\rho_{\mathrm{W}}(\infty) = P_{\mathrm{cl}}\rho_{\mathrm{cl}}(\infty) = \rho_{\mathrm{cl}}(\infty)$. This allows us to use highly developed classical periodic orbit theories to evaluate $\langle A\rangle_\infty$ [32]. These theories can be made very precise by the so called cycle expansion, which means that classical prime cycles p with similar actions are systematically combined to so-called pseudo orbits or cycles π. To expose these theories would be beyond the scope of this article. Let me just present the result for $\langle A\rangle_\infty$, explain how one uses it and show numerically that the agreement with *ab initio* quantum mechanical calculations is indeed very good.

Starting point for the practical use of the classical trace formulae is a list of prime cycles of the classical map, their stabilities, and topological lengths which has to be calculated numerically. Prime cycles are periodic orbits that can not be divided into smaller periodic orbits. The stabilities of a prime cycle are the (in the present context: two) stability eigenvalues, *i.e.* the eigenvalues of the Jacobian connected with the map from the starting point of the cycle to the last point before the cycle closes. Since phase space volume is not conserved for dissipative maps the product of the two eigenvalues usually does not equal unity, so we need to calculate always both of them. In the following, Λ_p will denote the product of all expanding eigenvalues (*i.e.* with absolute value larger than one) of a prime cycle p, and 1, if both are contracting. The topological length is for maps just the length in discrete time, with the convention that fixed points of f have topological length $n_p = 1$. Finally, we need the values A_p of the observable averaged along the prime cycles. Out of the prime cycles one has to construct all distinct non-repeating combinations $\{p_1, ..., p_k\}$ with a given total topological length $n_\pi = n_{p_1} + \ldots + n_{p_k}$. The other characteristics of the prime cycles are combined to corresponding quantities for the pseudo cycles as well, $A_\pi = A_{p_1} + \ldots + A_{p_k}$,

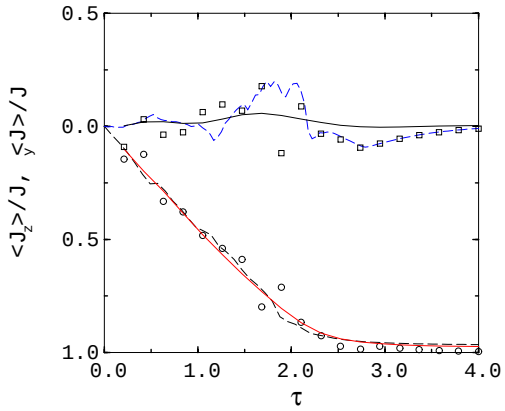

Fig. 2. Three different ways of calculating expectation values in the invariant state as function of the dissipation: quantum mechanically ($j = 20$, full lines), direct classical simulation (dashed lines), and classical periodic orbit formula with cycle expansion (circles: J_z/J, squares: J_y/J). The system parameters are $k = 8$ and $\beta = 2$.

and $\Lambda_\pi = \Lambda_{p_1} \cdot \ldots \cdot \Lambda_{p_k}$; and we define the pseudo-cycle weight $t_\pi = (-1)^{k+1}/|\Lambda_\pi|$, where k denotes the number of prime cycles involved. With all this, the expectation value of the classical observable A in the invariant state, $\langle A\rangle_\infty = \int A(x)\rho_{\mathrm{cl}}(x,\infty)\,\mathrm{d}x$ is given by [32]

$$\langle A\rangle_\infty = \frac{\sum_\pi A_\pi t_\pi}{\sum_\pi n_\pi t_\pi}\,. \tag{4.9}$$

The advantage of the cycle expansion is that the periodic orbit sum is truncated in a clever way, such that different contributions that would lead to a highly fluctuating behavior almost compensate. Figure 2 shows a comparison of exact quantum mechanical results for the observables J_z/J and J_y/J in the invariant state, compared to results from classical periodic orbit theory (4.9), as well as straightforward classical evaluations. The latter were obtained by iterating many randomly chosen initial phase space points and averaging over the generated trajectories. Whereas J_y fluctuates only slightly about the value zero suggested by the symmetry of the problem, J_z/J decreases from 0 at zero dissipation to -1 for strong dissipation as the strange attractor shrinks more and more towards towards the south pole of the Bloch sphere [2]. The figure shows that even with rather short orbits ($n_\pi \leq 4$) the quantum mechanical result is produced very well for both observables. The agreement improves as expected with larger values of J and becomes almost perfect when comparing the classical simulation with (4.9). Instead of ordering the cycles by topological length, I also tried stability ordering [33], but did not observe further significant improvement.

4.4 Correlation functions

The discrete time correlation function $K(N_2, N_1)$ between two observables A and B with respect to an initial density

matrix $\rho(0)$ is defined as [34]

$$K(N_2,N_1) = \langle B(N_2)A(N_1)\rangle_0 = \mathrm{tr}\left(BP^{N_2}AP^{N_1}\rho(0)\right). \tag{4.10}$$

This function has typically a real and an imaginary part. The latter is connected to the Fourier transform with respect to frequency of a linear susceptibility (see *e.g.* [17]). Here I show how the real part of $K(N_2,N_1)$ can be calculated semiclassically.

Starting point is the observation that in the above expression (4.10) $AP^{N_1}\rho(0) = A\rho(N_1)$ enters in the same way, as the initial density matrix $\rho(0)$ enters in the calculation of $\langle A(N)\rangle$ (Eq. (4.6)). In fact, formally $K(N_2,N_1)$ is nothing but the expectation value of B with respect to the N_2 times propagated "density matrix" $\rho'(N_1) \equiv A\rho(N_1)$. Note that $\rho'(N_1)$ is not really a density matrix, in general not even a Hermitian operator. However, in the derivation of expectation values the special properties of a density matrix (besides hermiticity also positivity, and trace equal unity) did not enter at any point. The only thing that did matter was that the density matrix had to have a smooth Wigner transform. This I will suppose as well about the Wigner transform $\rho'_{\mathrm{W}}(N_1)$ of $\rho'(N_1)$. Later on we will see that the assumption is justified if the initial density matrix $\rho(0)$ has a smooth Wigner transform and the Weyl symbol A_{W} a smooth classical limit. So let us introduce a Wigner transform $\rho'_{\mathrm{W}}(x,N_1)$ in complete analogy as for any density matrix (3.6),

$$\rho'_{\mathrm{W}}(\mu,\phi) = \frac{J^2}{\pi}\int \mathrm{d}\eta\, \mathrm{e}^{2\mathrm{i}J\eta\phi}\rho'(\mu,\eta,N_1) \tag{4.11}$$

and then use (4.7) to express the correlation function $K(N_2,N_1)$ as

$$K(N_2,N_1) = \int \mathrm{d}x\,\mathrm{d}y\, B_{\mathrm{cl}}(y)P_{\mathrm{cl}}^{N_2}\rho'_{\mathrm{W}}(x,N_1). \tag{4.12}$$

Now I write $\rho'(\mu,\eta,N_1) = \langle m+k|A\rho(N_1)|m-k\rangle$ in (4.11) as

$$\rho'(\mu,\eta,N_1) = J\int \mathrm{d}\lambda \sum_{n=-\infty}^{\infty} \langle m+k|A|J\lambda\rangle \times \langle J\lambda|\rho(N_1)|m-k\rangle \mathrm{e}^{\mathrm{i}J2\pi n\lambda}, \tag{4.13}$$

where I have introduced a factor unity with $l = J\lambda$ as summation variable and then changed the sum to an integral over l by Poisson summation. In terms of the corresponding Weyl symbol and Wigner function we have

$$\langle m+k|A|l\rangle = \frac{1}{2\pi}\int \mathrm{d}\phi_1 \exp\left(-\mathrm{i}J(\mu+\eta-\lambda)\phi_1\right) \times A_{\mathrm{W}}\left(\frac{\mu+\eta+\lambda}{2},\phi_1\right), \tag{4.14}$$

$$\langle l|\rho(N_1)|m-k\rangle = \frac{1}{J}\int \mathrm{d}\phi_2 \exp\left(-\mathrm{i}J(\lambda-\mu+\eta)\phi_2\right) \times \rho_{\mathrm{W}}\left(\frac{\lambda+\mu-\eta}{2},\phi_2,N_1\right). \tag{4.15}$$

If we insert the last two equations into (4.13) and the resulting one into (4.11) we are lead to

$$\rho'_{\mathrm{W}}(\mu,\phi,N_1) = \frac{J^2}{2\pi^2}\sum_{n=-\infty}^{\infty}\int \mathrm{d}\lambda\,\mathrm{d}\eta\,\mathrm{d}\phi_1\,\mathrm{d}\phi_2\, A_{\mathrm{W}}\left(\frac{\mu+\eta+\lambda}{2},\phi_1\right) \times \rho_{\mathrm{W}}(\frac{\lambda+\mu-\eta}{2},\phi_2,N_1)\exp(\mathrm{i}JH(\lambda,\eta,\phi_1,\phi_2)), \tag{4.16}$$

with an exponent H given by

$$H(\lambda,\eta,\phi_1,\phi_2) = 2\eta\phi - (\mu+\eta-\lambda)\phi_1 + (\mu-\eta-\lambda)\phi_2 + 2\pi n\lambda. \tag{4.17}$$

Integration by SPA leads to the saddle point equations

$$\partial_\eta H = 2\phi - \phi_1 - \phi_2 = 0 \tag{4.18}$$

$$\partial_\lambda H = \phi_1 - \phi_2 + 2\pi n = 0 \tag{4.19}$$

$$\partial_{\phi_1} H = -(\mu+\eta-\lambda) = 0 \tag{4.20}$$

$$\partial_{\phi_2} H = \mu-\eta-\lambda = 0. \tag{4.21}$$

The second one gives immediately $\phi_1 = \phi_2 \,\mathrm{mod} 2\pi$; and if we restrict ϕ_1,ϕ_2 as before to a 2π interval, we have $n = 0$ and $\phi_1 = \phi_2 = \phi$ from (4.18). The last two equations give $\eta = 0$ and $\lambda = \mu$. The value of H at the saddle point is zero and one easily checks that the determinant of second derivatives gives 4. Putting all pieces of the SPA together we obtain

$$\rho'_{\mathrm{W}}(x,N_1) = A_{\mathrm{W}}(x)\rho_{\mathrm{W}}(x,N_1). \tag{4.22}$$

This means that $\rho'_{\mathrm{W}}(x,N_1)$ is smooth if $A_{\mathrm{W}}(x)$ and $\rho_{\mathrm{W}}(x,N_1)$ are smooth. If we remember that to lowest order in $1/J$ the Weyl symbols A_{W} and B_{W} are just the classical observables A and B, we obtain from (4.10) the final result

$$K(N_2,N_1) = \int \mathrm{d}x\, B(f^{N_2}(x))A(x)\rho_{\mathrm{W}}(x,N_1). \tag{4.23}$$

So semiclassically, the correlation function has the same structure as a classical correlation function with respect to a classical initial phase space density $\rho_{\mathrm{cl}}(x)$,

$$K_{\mathrm{cl}}(N_2,N_1) = \int \mathrm{d}x\, B(f^{N_2}(x))A(x)\rho_{\mathrm{cl}}(x,N_1). \tag{4.24}$$

The only quantum mechanical ingredient is the Wigner function after N_1 steps. We have the same kind of hybrid classical-quantum formula as for expectation values. And as for expectation values, in the limit of large N_1 and with $N_2 - N_1 = N$ kept fixed, the quantum mechanical correlation function approaches its classical value, as $\rho_{\mathrm{W}}(x,N_1)$ tends to the smeared out classical invariant state $\rho_{\mathrm{cl}}(x,\infty)$. Nevertheless, as pointed out in the context of expectation values, $\rho_{\mathrm{W}}(x,N_1)$ can describe very non-classical states as for instance Schrödinger cat states [31].

Remarkable about (4.23) is also the fact that the expression is always real. We can trace this back to the

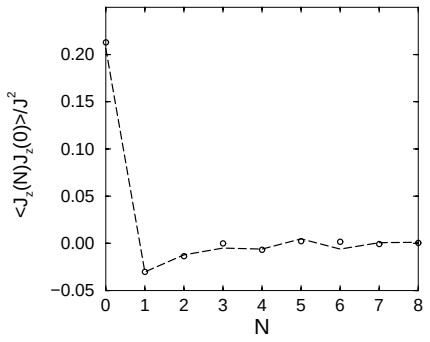

Fig. 3. The correlation function $\langle J_z(N)J_z(0)\rangle/J^2$ in the invariant state as function of N for $k = 8.0$, $\beta = 2.0$, $\tau = 1.0$. The real part of the quantum mechanical function ($j = 40$, circles) agrees well with the classical result (dashed line). For clarity the classical result has been plotted for continuous N, even though it is only defined for integer N.

realness of $\rho'_{\rm W}$ in (4.22). Since $A\rho(N_1)$ is not necessarily Hermitian, there would be no need for $\rho'_{\rm W}$ to be real. However, note that $A\rho(N_1)$ would be Hermitian, if A and $\rho(N_1)$ commuted. Since they do commute classically, the commutator must be of order $1/J$, and the imaginary part in $K(N_2, N_1)$ is therefore always at least of one order in $1/J$ smaller than the real part.

If $N_1 \to \infty$ (with $N = N_2 - N_1$ fixed) or if $\rho(x, 0)$ is chosen as the invariant density matrix so that $K(N + N_1, N_1)_{N_1\to\infty} \equiv K(N) = K_{\rm cl}(N) \equiv K_{\rm cl}(N + N_1, N_1)_{N_1\to\infty}$, we can use classical periodic orbit theory to calculate the quantum mechanical correlation function [35]. The use of the theory is completely analogous to the case of expectation values. In fact, the classical correlation function is nothing but the expectation value of $B(N)A(0)$ in the invariant state $\rho_{\rm cl}(\infty)$, so that in (4.9) we just insert for A_p the variable $B(N)A(0)$ averaged along the prime cycle p. The practical evaluation of $K(N)$ *via* the periodic orbit formula is, however, handicapped by the fact that for $K(N)$ one should have at least prime cycles of the length N. Finding all of these for large N is a difficult numerical problem, and hindered in our example additionally by the fact that we do not have a symbolic dynamics for the dissipative kicked top. Nevertheless, Figure 3 shows that at least the classical result and the real part of the quantum mechanical correlation function $\langle J_z(N)J_z(0)\rangle$ agree rather well.

5 Conclusions

In this article I have shown that in the presence of a small amount of dissipation ($\tau > 1/J$, where $J \to \infty$ in the classical limit), the propagator of Wigner functions that are smooth on a scale $1/J$ agrees up to order $1/J$ corrections with the classical Frobenius-Perron propagator of the phase space density. From this key result a number of important consequences followed. First, it allowed in a much simpler way than before to prove ${\rm tr}P^N = {\rm tr}P^N_{\rm cl}$ for fixed N as $J \to \infty$. Second, it gave rise to compact semiclassical formulae for expectation values and correlation functions of observables. Basically, expectation values and correlation functions of observables can be evaluated with hybrid classical-quantum formulae, with the only remainder of quantum mechanics lying in the initial condition, *i.e.* the initial Wigner function has to be used to average over phase space instead of a classical phase space density. If one iterates the map sufficiently many times, the Wigner function approaches the smeared out classical strange attractor and from then on the difference between quantum mechanics and classical mechanics is no longer visible in expectation values or correlation functions. Powerful classical periodic orbit theories can be applied to address questions in quantum mechanics. Another way to obtain classical results is to start from the very beginning with a Wigner function that is a classical phase space density.

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